The Yoneda Lemma

Theorem [The Yoneda Lemma] *For any functor* $F : \mathcal{C}^{op} \to \mathbf{Set}$ *and any object* X *in* $\mathcal{C}$*, natural transformations* $\mathrm{Hom}(-, X) \to F(-)$ *are in bijection with elements in* $F(X)$*; i.e.,*

$$\mathrm{Nat}(\mathrm{Hom}(-, X), F(-)) \cong F(X).$$

The intuition to have in mind here is that we can determine an object completely by testing all possible maps into it; i.e., in a category, $X \cong Y$ if and only if $\mathrm{Hom}(-, X) \cong \mathrm{Hom}(-, Y)$. (Indeed, this is a corollary we will see later.) We have an intuition for this already (and should because the Yoneda Lemma is ubiquitous throughout mathematics). As an example, many homeomorphism invariants in topology are of this form:

- X and Y have the same cardinality if and only if $\mathrm{Hom}(\{*\}, X) \cong \mathrm{Hom}(\{*\}, Y)$ for $\{*\}$ the point space.
- X and Y have the same path connected components if and only if $\mathrm{Hom}(I, X) \cong \mathrm{Hom}(I, Y)$ for $I \subseteq \mathbf{R}$ an interval.
- X and Y have the same fundamental group if and only if $\mathrm{Hom}(S^1, X) \cong \mathrm{Hom}(S^1, Y)$[1].
- X and Y have the same higher homotopy groups if and only if $\mathrm{Hom}(S^n, X) \cong \mathrm{Hom}(S^n, Y)$[2].

We will also explore some interesting applications of Yoneda after its proof.

Proof of the Yoneda Lemma. To show $\mathrm{Nat}(\mathrm{Hom}(-, X), F(-)) \cong F(X)$, we start with an element $t \in F(X)$, noting $F(X)$ is a set as $F : \mathcal{C}^{op} \to \mathbf{Set}$. We must produce a natural transformation $\mathrm{Hom}(-, X) \to F(-)$. Recall that a natural transformation $\eta : G \to H$ for functors $G, H : \mathcal{A} \to \mathcal{B}$ is a collection of morphisms $\eta_A : G(A) \to H(A)$ for all A objects of $\mathcal{A}$ such that for any arrow $f : A \to B$ in $\mathcal{A}$, we have commutativity of

$$\begin{array}{ccc} G(A) & \xrightarrow{G(f)} & G(B) \\ {\scriptstyle \eta_A}\downarrow & & \downarrow{\scriptstyle \eta_B} \\ H(A) & \xrightarrow[H(f)]{} & H(B) \end{array}$$

To produce a natural transformation, we need to describe $\eta_A : \mathrm{Hom}(A, X) \to F(A)$ for an arbitrary A, and then check that the requisite diagram commutes. We declare $\eta_A(\varphi : A \to X)$ to be $(F\varphi)(t)$. Since $\varphi : A \to X$ and F is contravariant, $F\varphi : F(X) \to F(A)$, and $t \in F(X) \mapsto (F\varphi)(t) \in F(A)$. Does the requisite diagram commute? The diagram is is, for $f : A \to B$,

$$\begin{array}{ccc} \mathrm{Hom}(B, X) & \xrightarrow{\mathrm{Hom}(f, X)} & \mathrm{Hom}(A, X) \\ {\scriptstyle \eta_B}\downarrow & & \downarrow{\scriptstyle \eta_A} \\ F(B) & \xrightarrow[Ff]{} & F(A) \end{array}$$

Taking a map $\psi : B \to X \in \mathrm{Hom}(B, X)$, we see that

$$\begin{array}{ccccc} \psi & \longmapsto & & \longrightarrow & \psi f \\ \downarrow & \mathrm{Hom}(B, X) & \xrightarrow{\mathrm{Hom}(f, X)} & \mathrm{Hom}(A, X) & \downarrow \\ & {\scriptstyle \eta_B}\downarrow & & \downarrow{\scriptstyle \eta_A} & \\ \downarrow & F(B) & \xrightarrow[Ff]{} & F(A) & \downarrow \\ (F\psi)(t) & \longmapsto & & \longrightarrow & (Ff \circ F\psi)(t) = F(\psi f)(t) \end{array}$$

[1] sorta
[2] sorta

Now, we need to take a natural transformation $\eta : \mathrm{Hom}(-, X) \to F(-)$ and produce an element $t \in F(X)$. To do this in a compatible way with the work above, the thing to do must be to map η to $\eta_X(\mathrm{id}_X)$. See that η_X takes $\mathrm{id}_X : X \to X$ and maps it to something in $F(X)$; call its image $t = \eta_X(\mathrm{id}_X)$.

To see that these assignments are inverses of each other, observe that

$$\begin{aligned} t &\mapsto \eta \text{ such that } \eta_A(\varphi : A \to X) = F(\varphi)(t) \\ &\mapsto \eta_X(\mathrm{id}_X) = F(\mathrm{id}_X)(t) = \mathrm{id}_{F(X)}(t) = t, \text{ and} \\ \eta &\mapsto \eta_X(\mathrm{id}_X) \\ &\mapsto \mu \text{ such that } \mu_A(\psi : A \to X) = F(\psi)(\eta_X(\mathrm{id}_X)), \text{ and also} \\ &\quad \eta \text{ satisfies } \eta_A(\psi : A \to X) = F(\psi)(\eta_X(\mathrm{id}_X)), \end{aligned}$$

since we claim η is uniquely determined by $\eta_X(\mathrm{id}_X)$. To see this, let $\psi : A \to X$ and observe the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(X, X) & \xrightarrow{\mathrm{Hom}(\psi, X)} & \mathrm{Hom}(A, X) \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_A} \\ F(X) & \xrightarrow[F\psi]{} & F(A) \end{array}$$

This diagram commutes because η is a natural transformation. Observe that if we chase the element $\mathrm{id}_X \in \mathrm{Hom}(X, X)$, we see that

$$\begin{array}{ccccc} \mathrm{id}_X & \longmapsto & & \longmapsto & \mathrm{id}_X \circ \psi = \psi \\ \big\downarrow & \mathrm{Hom}(X, X) & \xrightarrow{\mathrm{Hom}(\psi, X)} & \mathrm{Hom}(A, X) & \big\downarrow \\ \big\downarrow & {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_A} & \big\downarrow \\ \big\downarrow & F(X) & \xrightarrow[F\psi]{} & F(A) & \big\downarrow \\ \eta_X(\mathrm{id}_X) & \longmapsto & F(\psi)(\eta_X(\mathrm{id}_X)) & = & \eta_A(\psi) \end{array}$$

Hence the bijection $\mathrm{Nat}(\mathrm{Hom}(-, X), F(-)) \cong F(X)$. □

Let's explore the corollaries which are used to understand Yoneda.

Corollary 1 *The Yoneda embedding, $Y : \mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{op}}$, is a fully faithful functor.*

Proof. Recall the notation: $\mathbf{Set}^{\mathcal{C}^{op}}$ is the category of functors $\mathcal{C}^{op} \to \mathbf{Set}$ with arrows natural transformations between functors. The Yoneda embedding takes A in $\mathcal{C}$ and sends it to $Y(A) = \mathrm{Hom}(-, A)$, and takes $\varphi : A \to B$ in $\mathcal{C}$ and sends it to $Y(\varphi) = \varphi_* : \mathrm{Hom}(-, A) \to \mathrm{Hom}(-, B)$, which we know to compute as $\varphi_* = \varphi \circ -$. To show Y is fully faithful, we must show that the assignment

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathbf{Set}^{\mathcal{C}^{op}}}(Y(A), Y(B))$$

given by $\varphi \mapsto \varphi_*$ is injective (faithful) and surjective (full).

To see that the assignment is injective, notice that if φ and ψ mapping $A \to B$ are distinct, then φ_* and ψ_* are distinct, since we may detect so by realizing

$$\varphi_*(\mathrm{id}_A) = \varphi \circ \mathrm{id}_A = \varphi \neq \psi = \psi \circ \mathrm{id}_A = \psi_*(\mathrm{id}_A).$$

To see that the assignment is surjective, let $\eta : \mathrm{Hom}(-, A) \to \mathrm{Hom}(-, B) \in \mathrm{Hom}_{\mathbf{Set}^{\mathcal{C}^{op}}}(Y(A), Y(B))$. We need to show that η is of the form φ_* for some $\varphi : A \to B$. Explicitly, for $\varphi_* = \eta$, we must show that for every object C in $\mathcal{C}$ and arrow $\psi : C \to A$, we have $\eta_C(\psi) = \varphi_*(\psi) = \varphi \circ \psi$. We claim the map $\varphi : A \to B$ given by $\varphi = \eta_A(\mathrm{id}_A)$ does the job. Indeed, observe that since η is a natural transformation, by definition, for any objects C and D in $\mathcal{C}$ and any arrow $\psi : C \to D$,

$$\begin{CD}
\mathrm{Hom}(D,A) @>{\psi^*}>> \mathrm{Hom}(C,A) \\
@V{\eta_D}VV @VV{\eta_C}V \\
\mathrm{Hom}(D,B) @>>{\psi^*}> \mathrm{Hom}(C,B)
\end{CD}$$

for $\psi^* = - \circ \psi : \mathrm{Hom}(D, -) \to \mathrm{Hom}(C, -)$. Choosing $D = A$ and chasing $\mathrm{id}_A \in \mathrm{Hom}(A, A)$, we see that

$$\begin{array}{ccccc}
\mathrm{id}_A & \longmapsto & & \longrightarrow & \mathrm{id}_A \circ \psi = \psi \\
\big\downarrow & \mathrm{Hom}(A,A) & \xrightarrow{\psi^*} & \mathrm{Hom}(C,A) & \big\downarrow \\
 & \Big\downarrow{\scriptstyle \eta_A} & & \Big\downarrow{\scriptstyle \eta_C} & \\
 & \mathrm{Hom}(A,B) & \xrightarrow[\psi^*]{} & \mathrm{Hom}(C,B) & \\
\eta_A(\mathrm{id}_A) & \longmapsto & & \longrightarrow & \eta_A(\mathrm{id}_A) \circ \psi = \eta_C(\psi)
\end{array}$$

Now it is clear that if $\varphi = \eta_A(\mathrm{id}_A)$, then for all C in $\mathcal{C}$ and $\psi : C \to A$, we observe $\eta_C(\psi) = \varphi \circ \psi$, as desired. □

In succinct terms, an object X is the same as its representable functor $\mathrm{Hom}(-, X)$.

Corollary 2 *$A \cong B$ if and only if $\mathrm{Hom}(-, A) \cong \mathrm{Hom}(-, B)$.*

Proof. Since $Y(A) = \mathrm{Hom}(-, A)$ is a functor, if $A \cong B$, then $\mathrm{Hom}(-, A) \cong \mathrm{Hom}(-, B)$.

On the other hand, if $Y(A) \cong Y(B)$, then we claim since Y is fully faithful, that implies $A \cong B$. Indeed, observe:

> **Lemma** *Let F be a fully faithful functor. If $F(A) \cong F(B)$, then $A \cong B$.*
>
> *Proof.* If $f : F(A) \to F(B)$ is an isomorphism with inverse f^{-1}, then as F is fully faithful, there exists a unique $\varphi : A \to B$ such that $F\varphi = f$. Similarly, there exists a unique $\psi : B \to A$ such that $F\psi = f^{-1}$. Thus,
>
> $$\mathrm{id}_{F(A)} = f^{-1} f = F\psi F\varphi = F(\psi\varphi).$$
>
> As F is fully faithful, $\psi\varphi = \mathrm{id}_A$, since $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$. Similarly,
>
> $$\mathrm{id}_{F(B)} = f f^{-1} = F\varphi F\psi = F(\varphi\psi),$$
>
> and $\varphi\psi = \mathrm{id}_B$. Thus, φ is an isomorphism with inverse ψ. □

Hence, $A \cong B$, as desired, and the proof is complete. □

We now explore some applications and/or examples of the Yoneda Lemma.

Example 3 [Cayley's Theorem] *Every group G is isomorphic to a subgroup of the symmetric group acting on G, S_G.*

Proof. We can express a group G as a category $\mathcal{G}$ with one object $\bullet$ and whose arrows are isomorphisms corresponding to the group elements. Indeed, it is an exercise to show that the category axioms correspond to the group axioms:

- Associativity of group elements is associativity of arrows.
- Existence of an identity $e \in G$ is existence of an identity arrow $\mathrm{id}_\bullet$.
- Existence of inverse elements is existence of inverse arrows, as each arrow is an isomorphism.

An object in $\mathbf{Set}^{\mathcal{G}^{op}}$, i.e., a functor $\mathcal{G}^{op} \to \mathbf{Set}$, sends $\bullet$ to a set X and sends an arrow (group element g) to $X \to X$ defined by right multiplication by g. In particular, choosing $F(-) = \mathrm{Hom}(-, \bullet) \in \mathbf{Set}^{\mathcal{G}^{op}}$, the Yoneda Lemma says that

$$\mathrm{Nat}(\mathrm{Hom}(-, \bullet), F(-)) \cong F(\bullet)$$
$$\mathrm{Nat}(\mathrm{Hom}(-, \bullet), \mathrm{Hom}(-, \bullet)) \cong \mathrm{Hom}(\bullet, \bullet).$$

The right side is, by definition, elements of the group G. Decoding the left side, we see that natural transformations $\mathrm{Hom}(-, \bullet) \to \mathrm{Hom}(-, \bullet)$ are some subset of G-equivariant functions $G \to G$. In particular, they are the G-equivariant functions that are constructed from the elements of G, by **Corollary 1**. In other words, $\mathrm{Nat}(\mathrm{Hom}(-, \bullet), \mathrm{Hom}(-, \bullet))$ is the set of functions $f_g : G \to G$ defined by $f_g(x) = xg$; i.e., the automorphisms of G that arise from g-multiplication. Hence $\mathrm{Nat}(\mathrm{Hom}(-, \bullet), \mathrm{Hom}(-, \bullet))$ is a subgroup of S_G, all permutations on G, and by the Yoneda Lemma, is isomorphic to $\mathrm{Hom}(\bullet, \bullet) \cong G$. □

Example 4 [Dedekind cuts] *The real numbers can be constructed using Yoneda.*

Proof. Let $\mathbf{Q}$ be conflated with the poset category whose objects are rational numbers and whose arrows are given by the ordering $\leq$. (It is once again an easy exercise to see the poset axioms for $\leq$ are the category axioms for $\to$.) The Yoneda embedding

$$Y : \mathbf{Q} \to \mathbf{Set}^{\mathbf{Q}^{op}}$$

is given by $Y(-) = \mathrm{Hom}(-, q)$. In fact, we may narrow our focus, and rather than viewing $\mathbf{Q}$ as "enriched over **Set**," we may view it as "enriched over **2**," where **2** is the category with objects 0 and 1 and nontrivial arrow $0 \to 1$. Recall that a category $\mathcal{C}$ is enriched over $\mathcal{D}$ if for all objects A and B in $\mathcal{C}$, $\mathrm{Hom}(A, B)$ is in $\mathcal{D}$. In layman's terms, often Hom-objects carry structure other than sets; they may be abelian groups, as in the case of **Ab**-categories, for instance.

Hence the Yoneda embedding can be thought of as

$$Y : \mathbf{Q} \to \mathbf{2}^{\mathbf{Q}^{op}},$$

where the Hom-object $\mathrm{Hom}(r, s) = 1$ if $r \leq s$, and 0 if not; i.e., we simply indicate whether or not the inequality is true.

The set $\mathrm{Hom}(-, q) = 1$ is a Dedekind cut; the collection of all arrows $x \to q$ is the set $\{x \in \mathbf{Q} \mid x \leq q\}$. By **Corollary 1**, $\mathrm{Hom}(-, q)$ embeds in a full and faithful manner; i.e., we can define $\mathbf{R}$ to be the full subcategory of $\mathbf{2}^{\mathbf{Q}^{op}}$ for which all functors $\mathbf{Q}^{op} \to \mathbf{2}$ are non-constant and cocontinuous, i.e., preserve colimits. In a poset, a colimit is just a supremum, so $\mathbf{R}$ is the collection of suprema of sets $\{x < r\}$; i.e., the classical Dedekind construction of $\mathbf{R}$. □